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LETTER TO THE EDITOR

Canonical entanglement for two indistinguishable particles

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Abstract
We determine the degree of entanglement for two indistinguishable particles based on the tensor product structure, which is a framework for emphasizing entanglement founded on observational quantities. Our theory connects the canonical entanglement and entanglement based on occupation number for two fermions and for two bosons and shows that the entanglement measure, based on linear entropy, is closely related to the correlation measure for both the bosonic and fermionic cases.

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Introduction

Entanglement and the indistinguishability of particles are two remarkable features of quantum mechanics, yet combining the two in order to obtain a meaningful quantification of entanglement for indistinguishable particles is challenging. Several definitions of correlation and entanglement have been introduced [1–7], which both illustrate the choices that are available in quantifying entanglement and underpin the ambiguities about what constitutes the best measure of entanglement.

If entanglement is to be associated with observational properties that can overcome the indistinguishability of the particles, then the tensor product structure (TPS) [8] plays a key role. In fact entanglement itself is a consequence of the TPS and the superposition principle. However, for systems consisting of many indistinguishable particles such as bosons and fermions, the TPS is rather subtle. Approaches to quantifying entanglement can be subdivided into two main categories: one based on the canonical decomposition [1–3] and the other based on the occupation-number representation [8, 4]. Here we employ both approaches to determine entanglement for two indistinguishable particles. Our approach is built on two steps: (i) to use the canonical decomposition to obtain the canonical form of the two-particle states and (ii) to define entanglement from the canonical form of the state by the approach based on the occupation-number representation. We refer to this form of entanglement for indistinguishable particles as *canonical entanglement*. Although large numbers of particles can be considered, the method is given for two particles, which can be used as a primitive to treat multiple particles.

Tensor product structure for states of two indistinguishable particles

A pure state of two particles can be written as

$$|\Psi\rangle = \sum_{i,j=1}^{M} \Omega_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle, \tag{1}$$

with $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$ creation operators for modes i and j, respectively, and $|0\rangle$ being the vacuum state (no particles at all). For the case of fermions, Ω is an antisymmetric matrix and the creation operators are fermionic, while for the case of bosons, Ω is a symmetric matrix and the creation operators are bosonic.

For the case of fermions, we assume that there is an even number of modes and set $M \equiv 2N$; for the bosonic case the number of modes is not important, and we will assign $M = N$. The canonical form for the two-particle states can be obtained with the help of the singular value decomposition (SVD) [2]. For any antisymmetric $2N \times 2N$ matrix $\Omega^{\mathrm{A}} \neq 0$, there exists a unitary operator U^{A} such that $\Omega^{\mathrm{A}} = U^{\mathrm{A}} Y^{\mathrm{A}} (U^{\mathrm{A}})^T$, for $Y^{\mathrm{A}} \equiv \mathrm{diag}\big[Y_1^{\mathrm{A}}, \ldots, Y_N^{\mathrm{A}}\big]$ block diagonal with blocks

$$Y_i^{\mathrm{A}} = \begin{pmatrix} 0 & y_i^{\mathrm{A}} \\ -y_i^{\mathrm{A}} & 0 \end{pmatrix}, \tag{2}$$

and y_i^{A} may be zero [1]. This decomposition is unique and yields the fermionic state

$$|\Psi_F\rangle = \sum_{k=1}^{N} 2y_k \hat{a}_{2k-1}^{\prime\dagger}\hat{a}_{2k}^{\prime\dagger}|0\rangle = \sum_{k,l=1}^{M} Y_{kl}^{\mathrm{A}} \hat{a}_k^{\prime\dagger}\hat{a}_l^{\prime\dagger}|0\rangle, \tag{3}$$

for $\hat{a}_k^{\prime\dagger} \equiv \sum_{i=1}^N U_{ik}\hat{a}_i^\dagger$ new fermionic operators. The above form is the canonical representation [9, 10] so we refer to the state (3) as being represented in canonical form. The uniqueness of the SVD ensures that the canonical modes themselves are unique.

Now that we have the canonical form of the two-particle fermion state, we can impose the TPS [8] first by establishing the following operators:

$$\begin{aligned} &\hat{\sigma}_{k+} = \hat{a}_{2k-1}^{\prime\dagger}\hat{a}_{2k}^{\prime\dagger}, \hat{\sigma}_{k-} = \hat{a}'_{2k-1}\hat{a}'_{2k}, \\ &\hat{\sigma}_{kz} = \tfrac{1}{2}\big(\hat{a}_{2k-1}^{\prime\dagger}\hat{a}'_{2k-1} + \hat{a}_{2k}^{\prime\dagger}\hat{a}'_{2k} - 1\big), \end{aligned} \tag{4}$$

which obey $su(2)$ commutation relations. As operators with different subscripts k commute with each other, the state effectively comprises distinguishable particles. Furthermore, the state can be regarded as an N-qubit system with the kth qubit state given by the vacuum state $|0\rangle_k$ and $|1\rangle_k = \hat{a}_{2k-1}^{\prime\dagger}\hat{a}_{2k}^{\prime\dagger}|0\rangle$; the TPS is now evident:

$$|\Psi_F\rangle = \sum_{k=1}^{N} 2y_1^{\mathrm{A}}|100\ldots0\rangle + 2y_2^{\mathrm{A}}|010\ldots0\rangle + \cdots + 2y_N^{\mathrm{A}}|000\ldots1\rangle. \tag{5}$$

Further discussions of relevant mappings from fermions to qubits can be found in [11].

For two bosons, Ω in equation (1) is a symmetric complex matrix, and $\hat{a}_i^\dagger$ and $\hat{a}_i$ are bosonic creation and annihilation operators. For any symmetric $N \times N$ matrix $\Omega^{\mathrm{S}} \neq 0$, there exists a unitary operator U^{S} such that $\Omega^{\mathrm{S}} = U^{\mathrm{S}} Y^{\mathrm{S}} (U^{\mathrm{S}})^T$, with $Y^{\mathrm{S}} = \mathrm{diag}\big[y_1^{\mathrm{S}}, \ldots, y_N^{\mathrm{S}}\big]$ and y_i^{S} possibly zero for some values of i. Applying this unique decomposition to the bosonic state $|\Psi_B\rangle$ obtained from (1) yields

$$|\Psi_B\rangle = \sum_{k=1}^{N} y_k^{\mathrm{S}} \hat{a}_k^{\prime\dagger 2} |0\rangle, = \sum_{k,l=1}^{N} Y_{kl}^{\mathrm{S}} \hat{a}_k^{\prime\dagger} \hat{a}_i^{\prime\dagger} |0\rangle, \tag{6}$$

where

$$\hat{a}_k^{\prime\dagger} = \sum_{i=1}^{N} U_{ik}^{S} \hat{a}_i^{\dagger} \tag{7}$$

are new bosonic canonical creation operators. The TPS for the two-particle bosonic state is now clear:

$$|\Psi_B\rangle = \sum_{k=1}^{N} \sqrt{2} y_1^{\mathrm{S}} |200\ldots0\rangle + \sqrt{2} y_2^{\mathrm{S}} |020\ldots0\rangle + \cdots + \sqrt{2} y_N^{\mathrm{S}} |000\ldots2\rangle. \tag{8}$$

If we view the two-boson state $|2\rangle$ as a one-excitation state $|1\rangle$, this state can be regarded as a multiqubit state, and its entanglement is well defined.

Correlation measures and average entanglement

Paškauskas and You proposed a correlation measure based on the von Neumann entropy [2]. After deriving the above results, they first obtain the single-particle density matrix and then obtain the correlation measure determined by the von Neumann entropy for this reduced state. For both cases of two fermions and two bosons, the reduced density matrix is given by

$$\rho_{\nu\mu} = \frac{\mathrm{Tr}\big(\hat{\rho} \hat{a}_\mu^\dagger \hat{a}_\nu\big)}{\mathrm{Tr}\big(\hat{\rho} \sum_{\mu=1}^{M} \hat{a}_\mu^\dagger \hat{a}_\mu\big)} = 2(\Omega^\dagger \Omega)_{\mu\nu}, \tag{9}$$

with $M = 2N$ and $\Omega = \Omega^{\mathrm{A}}$ for fermions and $M = N$ and $\Omega = \Omega^{\mathrm{S}}$ for bosons.

The von Neumann entropy can be computed from the matrix elements of equation (9) to obtain

$$\begin{aligned} S &= -\mathrm{Tr}[\hat{\rho} \log_2(\hat{\rho})] \\ &= \begin{cases} -1 - 4\sum_{k=1}^{N} \big|y_k^{\mathrm{A}}\big|^2 \log_2\big(\big|\big(y_k^{\mathrm{A}}\big)^2\big|\big), & \text{for} \quad 2 \text{ fermions}, \\ -\sum_{k=1}^{N} \big(2\big|y_k^{\mathrm{S}}\big|^2\big) \log_2\big(2\big|\big(y_k^{\mathrm{S}}\big)^2\big|\big), & \text{for} \quad 2 \text{ bosons}. \end{cases} \end{aligned} \tag{10}$$

Paškauskas and You used the above von Neumann entropy to quantify the quantum correlations, and in the following we let $S_F(S_B)$ denote the entropy for fermions (bosons). For an uncorrelated state, the entropy $S_F = 1$ for two fermions, thus one encounters the curious situation that the entropy is not zero for an uncorrelated state [6]. As we will see shortly, the average entanglement given below vanishes for the uncorrelated state, and thus this problem is remedied. In contrast to the fermionic case, $S_B = 0$ does hold for an uncorrelated two-boson state.

Different entropies can be employed to quantify entanglement and correlations, and the Von Neumann entropy is one of them. We use it to quantify the entanglement between one qubit with the rest, namely, the Von Neumann entropy is employed to quantify both the quantum correlations and entanglement. From equation (5), the entanglement between the kth fermionic qubit and the rest is obtained as

$$E_{F,k} = h\big(4\big|y_k^{\mathrm{A}}\big|^2\big) \tag{11}$$

for

$$h(x) \equiv -x \log_2 x - (1-x)\log_2(1-x). \tag{12}$$

We employ the average entanglement to characterize the global entanglement properties; average entanglement has been employed effectively for studies of nonlinear inhomogeneous systems [12–15]. Thus, from equation (11), the average entanglement is given by

$$E_F = \frac{1}{N}\sum_{k=1}^{N} E_{F,k} = \frac{1}{N}\sum_{k=1}^{N} h\left(4\left|y_k^{\mathrm{A}}\right|^2\right). \tag{13}$$

For a non-entangled state, the entanglement measure $E_F = 0$. Moreover, E_F can be used to quantify the entanglement of two fermions. From equations (10) and (13), we find a relation between E_F and S_F, namely

$$E_F = \frac{1}{N}\left[S_F - 1 - \sum_{k=1}^{N}\left(1-4\left|y_k^{\mathrm{A}}\right|^2\right)\log_2\left(1-4\left|y_k^{\mathrm{A}}\right|^2\right)\right]. \tag{14}$$

The above equation shows that the entanglement measure E_F for two fermions is closely related to the correlation measure S_F.

For two bosons, from equation (8), the entanglement between the kth qubit and the remaining $N-1$ qubits is given by

$$E_{B,k} = h\left(2\left|y_k^{\mathrm{S}}\right|^2\right), \tag{15}$$

and the average entanglement is

$$\begin{aligned} E_B &= \frac{1}{N}\sum_{k=1}^{N} h\left(2\left|y_k^{\mathrm{S}}\right|^2\right) \\ &= \frac{1}{N}\left[S_B - \sum_{k=1}^{N}\left(1-2\left|y_k^{\mathrm{S}}\right|^2\right)\log_2\left(1-2\left|\left(y_k^{\mathrm{S}}\right)^2\right|\right)\right]. \end{aligned} \tag{16}$$

For a non-entangled state, $E_B = S_B = 0$. We see that our measures of entanglement for two indistinguishable particles are closely related to the correlation measures for both cases of bosons and fermions. To reveal a more direct connection between the canonical entanglement measure and the correlation measure, we next adopt the linear entropy to quantify entanglement and correlations. One merit of the linear entropy is that it is simpler to calculate and manipulate than the von Neumann entropy.

Linear entropy for the measure of entanglement

The linear entropy for a state $\hat{\rho}$ is defined as

$$E' \equiv 1 - \mathrm{Tr}(\hat{\rho}^2). \tag{17}$$

As we are considering pure states, we may employ either the linear entropy or the Von Neumann entropy to study the bipartite entanglement.

From equation (9), the two-fermion quantum correlation quantified by the linear entropy is

$$S_F' = 1 - 8\sum_{k=1}^{N} |y_k^{\mathrm{A}}|^4. \tag{18}$$

From equation (5), the entanglement between the kth qubit and the rest is given by

$$E'_{F,k} = 8\left(|y_k|^2 - \sum_{k=1}^{N} 4|y_k|^4\right). \tag{19}$$

Then, the average linear entropy is obtained as

$$E'_F = \frac{2}{N}\left(1 - \sum_{k=1}^{N} 16|y_k|^4\right). \tag{20}$$

We use the average linear entropy to quantify the entanglement of two fermions. From equations (18) and (20), we obtain

$$E'_F = \frac{2}{N}(2S'_F - 1). \tag{21}$$

Thus, the entanglement measure E'_F is proportional to the correlation measure S'_F up to an additive constant. This result is important as we can now claim that the correlation of two fermions considered by Paškauskas and You can be viewed as entanglement.

For two bosons, the correlation measure quantified by the linear entropy is

$$S'_B = 1 - 4\sum_{k=1}^{N} \left|y_k^{\mathrm{S}}\right|^4. \tag{22}$$

From equation (8), the average entanglement is given by

$$E'_B = \frac{2}{N}\left(1 - 4\sum_{k=1}^{N} \left|y_k^{\mathrm{S}}\right|^4\right). \tag{23}$$

It is evident that the two measures are connected by the following relation:

$$E'_B = \frac{2}{N} S'_B, \tag{24}$$

i.e. the entanglement measure E'_B is exactly proportional to the correlation measure S'_B. In other words, the entanglement and correlation measures are equivalent up to a multiplicative factor if we adopt the linear entropy to quantify them.

Conclusions

In conclusion, we have given entanglement measures of two indistinguishable particles, and both cases of bosons and fermions are considered. The approach here combines the advantages of the approach based on the canonical decomposition and another one based on the occupation-number basis. We also exploit the concept of average entanglement, characterizing the global entanglement properties of the system.

We compare the entanglement measure with the correlation measure, and find they are related. Specifically, we find that if we adopt linear entropy to quantify entanglement and correlation, the entanglement measure for two fermions is a linear function of corresponding correlation measure, and the entanglement measure for two bosons is equivalent to the correlation measure up to a multiplicative constant. The correlation of two fermions considered by Paškauskas and You can be viewed as entanglement, and this relationship, in turn, supports our choices of entanglement measures.

Although we restricted ourselves to the two-particle cases, our approach sheds new light on quantification of entanglement of indistinguishable many-body systems. We may study entanglement in many-fermion systems by appropriately pairing fermions. The TPS is the first premise of quantum entanglement, and thus we have to identify a TPS in indistinguishable systems in order to define entanglement. The various TPSs give rise to different entanglements. For instance, by mapping $2N$ fermion modes to $2N$ qubits [4], we identify another TPS. The entanglement via this TPS is obviously different from that via the TPS by pairing fermions in the above discussions. The various TPSs lead to different entanglements which may lead to ambiguities. However, this ambiguity is really an indication of the complexity of entanglement, associated with the variety of purposes for which entanglement is useful.

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